Transformations of discrete isothermic nets and discrete cmc-1 surfaces in hyperbolic space

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February 8, 2008

Summary. Using a quaternionic calculus, the Christoffel, Darboux, Goursat, and spectral transformations for discrete isothermic nets are described, with their interrelations. The Darboux and spectral transformations are used to define discrete analogs for cmc-1 surfaces in hyperbolic space and to obtain a discrete version of Bryant's Weierstrass type representation.

1. Introduction

There is a rich transformation theory for smooth isothermic surfaces that was mostly developed in classical times:

Christoffel (1867) showed that, to a given isothermic surface (in Euclidean 3-space), there is (locally) a second isothermic surface and a conformal point-to-point correspondence between the two surfaces such that the tangent planes in corresponding points are parallel — in fact, he showed that the existence of such a (non-trivial) transform *characterizes* isothermic surfaces [11];

Darboux (1899) found the existence of a $(1 + 3)$-parameter family of sphere congruences that are enveloped by a given isothermic surface, such that the induced map onto the second envelope preserves curvature lines and is conformal — this second envelope (transform) is then also isothermic [12];

Bianchi (1905) introduced a 1-parameter family of deformations for an isothermic surface — that he called the "T-transformation" — and analyzed the interrelations of the various transformations in terms of "permutability theorems" [2]. This last transformation seems to play a central role in the theory of isothermic surfaces: it relates to the characterization of isothermic surfaces as the only non-rigid surfaces in Möbius geometry [17] (cf.[10],[18]) as well as to the integrable system approach to isothermic surfaces via curved flats in the symmetric

space of point pairs [7] (cf.[13],[9]).

Discretizations for the Christoffel- and the Darboux-transformations have been given, using a discrete version of Christoffel's original formula and of the Riccati-type partial differential equation that we derived in [14]: for both ansatzes, it was crucial to have a Euclidean description of the transformation (even though, for the discrete analog of the Darboux-transformation, a Möbius invariant definition can be given — as it should be). The T-transformation is rather a transformation for Möbius equivalence classes of isothermic surfaces than for surfaces with a fixed position in space (in contrast to the Darboux transformation that incorporates the position of the surface); consistently, any desription should make use of the Möbius group. Thus, in order to introduce a discrete analog for the T-transformation, it seemed necessary to elaborate a suitable setup (section 2): a quaternionic approach[1] seems preferable over the classical formalism as no normalization problems occur.

With this setup, the transformation theory of discrete isothermic nets is elaborated (section 3) — entirely analogous to the transformations of (smooth) isothermic surfaces (cf.[15]): besides the T-transformation, a notion of "general Christoffel (C-) transformation" (cf.[9]) and a (generalized) "Goursat (G-) transformation" are introduced; also, a new definition for the Darboux (D-) transformation is given and shown to be equivalent to our previous definition [16]. Finally, the interrelations between the different transformations are discussed in terms of permutability theorems.

The transformations of isothermic surfaces already provided useful ansatzes to define discrete minimal (cf.[5]) and discrete cmc [16] nets. With the T-transformation available, we may use it to define discrete analogs for surfaces of constant mean curvature in other space forms: in case of cmc surfaces, the T-transformation becomes Lawson correspondence (cf.[13]) — and, in particular, it becomes the Umehara-Yamada perturbation in case of minimal surfaces in Euclidean space and cmc-1 surfaces in hyperbolic space (cf.[15]). Thus, discrete analogs for surfaces of constant mean curvature H in a space of constant curvature k, that can be obtained from cmc surfaces in Euclidean space via Lawson correspondence (i.e. $k + H^2 \geq 0$), can be defined as T-transforms of minimal or cmc nets in Euclidean space.

However, for cmc-1 surfaces in hyperbolic space, a more geometric approach is obtained via the Darboux transformation as it couples the surface with its hyperbolic Gauss map [15]: this approach will be used to introduce the notion of "horospherical nets" as an analog of cmc-1 surfaces in hyperbolic space, in section 4. The permutability theorems for various transformations will show that this ansatz is equivalent to the previously described one — this way, the notion of the "minimal cousin" of a horospherical net makes sense. Also, they will serve in obtaining a discrete version of Bryant's Weierstrass type representation; comparison with the other Weierstrass type representation in terms of

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[1] Or, in higher dimensions, an approach using Clifford algebras [8] (or, [4]).

the Darboux transformation, a discrete analog of the "dual transformation" for cmc-1 surfaces by Umehara and Yamada [21] naturally appears.

2. Preliminaries

Just as the geometry of the complex projective line $\mathbb{C}P^1 \cong S^2$ is the geometry of (orientation preserving) Möbius transformations of the conformal 2-sphere, 1-dimensional quaternionic projective geometry can be identified with the Möbius geometry of the conformal 4-sphere $S^4 \cong \mathbb{H}P^1 := \{v\mathbb{H} \mid v \in \mathbb{H}^2\}$ (cf.[13]). Here, we consider the space $\mathbb{H}^2$ of homogeneous coordinates as a *right* vector space over the quaternions. In this model, $PGl(2,\mathbb{H}) = Gl(2,\mathbb{H})/\mathbb{R}I$ acts by (orientation preserving) Möbius transformations on $\mathbb{H}P^1$ via $v\mathbb{H} \to (Mv)\mathbb{H}$, and 3-spheres appear as null cones of suitable quaternionic hermitian forms $s : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{H}$: equipped with the Minkowski product induced[2] by $-\det$ as a quadratic form the space $\mathbb{R}_1^6$ of quaternionic hermitian forms becomes the space of homogeneous coordinates for the classical model space $\mathbb{R}P^5$ of 4-dimensional Möbius geometry ([3], cf.[13]) with $PGl(2,\mathbb{H})$ acting via $(M,s) \to M \cdot s := s(M^{-1}., M^{-1}.)$.

To any quaternionic line $v\mathbb{H} \subset \mathbb{H}^2$ there is a unique line $\mathbb{H}\nu = (v\mathbb{H})^\perp \subset (\mathbb{H}^2)^*$ of quaternionic 1-forms annihilating $v\mathbb{H}$ — note that the dual space $(\mathbb{H}^2)^*$ is a *left* vector space over $\mathbb{H}$ since $\mathbb{H}^2$ is a right vector space. Thus, the conformal 4-sphere can be identified $(\mathbb{H}P^1)^\perp := \{\mathbb{H}\nu \mid \nu \in (\mathbb{H}^2)^*\} \cong \mathbb{H}P^1$. And, $PGl(2,\mathbb{H})$ acts on $(\mathbb{H}P^1)^\perp$ via $\mathbb{H}\nu \to \mathbb{H}(M \cdot \nu) = \mathbb{H}(\nu M^{-1})$.

Fixing homogeneous coordinates $s \in \mathbb{R}_1^6$ for a 3-sphere $S^3 \cong \mathbb{R}s \subset \mathbb{R}_1^6$ provides a canonical identification $\mathbb{H}^2 \leftrightarrow (\mathbb{H}^2)^*$ via $v \to s(v,.)$. Indeed, $v \to \sigma_v := s(v,.)$ is an anti-isomorphism as it is anti-linear and $\sigma_v = 0$ if and only if $v = 0$:

$$\begin{aligned} |s|^2 v &= e_1\{s(e_1,e_2)s(e_2,v) - s(e_1,v)s(e_2,e_2)\} \\ &+ e_2\{s(e_2,e_1)s(e_1,v) - s(e_2,v)s(e_1,e_1)\}. \end{aligned} \tag{1}$$

Moreover, for $v \in S^3$, i.e. $s(v,v) = 0$ and $v \neq 0$, $(v\mathbb{H})^\perp = \mathbb{H}\sigma_v$ since $\sigma_v \neq 0$ implies that $\sigma_v(w) \neq 0$ unless $w = v\lambda$. Consequently, the quaternionic hermitian form $s^* : (\mathbb{H}^2)^* \times (\mathbb{H}^2)^* \to \mathbb{H}$, $s^*(\sigma_v, \sigma_w) = s(v,w)$ defines S^3 in the $(\mathbb{H}P^1)^\perp$-model of S^4. The Möbius transformations of $S^3 \cong \mathbb{R}s$ are those fixing $\mathbb{R}s$, i.e. $\mathbb{R}(M^{-1} \cdot s) = \mathbb{R}s$.

The above identification $\mathbb{H}P^1 \cong (\mathbb{H}P^1)^\perp$ can be used to calculate the cross ratio [16] of four points in $\mathbb{H}P^1$, or in S^3, respectively:

Lemma (Cross ratio). *The cross ratio of four points $p_i \in \mathbb{H}P^1$, $i = 1,\ldots,4$, is given by* $[p_1,p_2,p_3,p_4] = \mathrm{Re}\,q + i\,|\mathrm{Im}\,q|$, *where*

$$q = (\nu_1 v_2)(\nu_3 v_2)^{-1}(\nu_3 v_4)(\nu_1 v_4)^{-1} \tag{2}$$

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[2] We set $\det s := s(e_1,e_1)s(e_2,e_2) - |s(e_1,e_2)|^2$ relative to a fixed basis (e_1,e_2) of $\mathbb{H}^2$ — a different choice of basis results in a rescaling of the Minkowski product: this ambiguity reflects the fact that the geometrically relevant space is the projective space $\mathbb{R}P^5$ with absolute quadric $\det = 0$.

with homogeneous coordinates $v_i \in I\!\!H^2$ and $\nu_i \in (I\!\!H^2)^*$ of $p_i \in I\!\!HP^1 \cong (I\!\!HP^1)^\perp$.

For four points $p_i \in S^3 \cong s$ their homogeneous coordinates $\nu_i = \sigma_{v_i} \simeq v_i$ in $(I\!\!H^2)^* \cong I\!\!H^2$ can be identified, so that

$$q = (s(v_1, v_2))(s(v_3, v_2))^{-1}(s(v_3, v_4))(s(v_1, v_4))^{-1}. \tag{3}$$

Obviously, (2) is independent of the choice of homogeneous coordinates, as well as it is invariant under Möbius transformations. The easiest way to see that (2) really gives the cross ratio of the four points is to use affine coordinates: without loss of generality $p_i \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix} I\!\!H \simeq \infty \in I\!\!HP^1$ so that one can choose $v_i = \begin{pmatrix} \mathfrak{p}_i \\ 1 \end{pmatrix}$ and $\nu_i = (1, -\mathfrak{p}_i)$ with suitable $\mathfrak{p}_i \in I\!\!H$; this way, one obtains the usual (cf.[16]) formula $q = (\mathfrak{p}_1 - \mathfrak{p}_2)(\mathfrak{p}_2 - \mathfrak{p}_3)^{-1}(\mathfrak{p}_3 - \mathfrak{p}_4)(\mathfrak{p}_4 - \mathfrak{p}_1)^{-1}$.

A more invariant way of describing the choice of affine coordinates is via stereographic projection: choosing homogeneous coordinates $v_0, v_\infty \in I\!\!H^2$ of two points $0, \infty \in I\!\!HP^1 \cong (I\!\!HP^1)^\perp$ their homogeneous coordinates $\nu_0, \nu_\infty \in (I\!\!H^2)^*$ can be canonically fixed by the condition $v_0\nu_\infty + v_\infty\nu_0 = id$, i.e. (v_0, v_∞) and (ν_0, ν_∞) are "pseudo dual" bases. Then, if $v \in I\!\!H^2$ and $\nu \in (I\!\!H^2)^*$ denote homogeneous coordinates of a point $\infty \neq p = I\!\!HP^1 \cong (I\!\!HP^1)^\perp$, $vI\!\!H = (v_\infty\mathfrak{p} + v_0)I\!\!H$ and $I\!\!H\nu = I\!\!H(\nu_0 - \mathfrak{p}\nu_\infty)$ with $\mathfrak{p} := (\nu_0 v)(\nu_\infty v)^{-1} = -(\nu v_\infty)^{-1}(\nu v_0)$ where the last equality follows from $\nu v = 0$. $\mathfrak{p} = \mathfrak{p}(p)$ will be called the "stereographic projection" of $p \in I\!\!HP^1$. Note that $\mathfrak{p}$ does not depend on the choice of homogeneous coordinates v, or ν, of p; a different choice of homogeneous coordinates for 0 and ∞, $\tilde{v}_0 = v_0 a_0$ and $\tilde{v}_\infty = v_\infty a_\infty^{-1}$ or, equivalently, $\tilde{\nu}_0 = a_\infty\nu_0$ and $\tilde{\nu}_\infty = a_0^{-1}\nu_\infty$, yields a stretch-rotation $\tilde{\mathfrak{p}} = a_\infty\mathfrak{p}a_0$ of the stereographic projection. Now, for two points $p_1, p_2 \in I\!\!HP^1 \cong (I\!\!HP^1)^\perp$, a simple calculation yields

$$(\nu_1 v_\infty)^{-1}(\nu_1 v_2)(\nu_\infty v_2)^{-1} = \mathfrak{p}_2 - \mathfrak{p}_1. \tag{4}$$

Restricting to points in a 3-sphere $S^3 \cong I\!\!Rs$, the identification $I\!\!H^2 \cong (I\!\!H^2)^*$ can be used to fix the relative scaling of homogeneous coordinates $v_0, v_\infty \in I\!\!H^2$ for $0, \infty \in S^3 \subset I\!\!HP^1$, via $s(v_0, v_\infty) = 1$; then, $|s| = 1$ with respect to (v_0, v_∞) as a basis of $I\!\!H^2$, and (1) reduces to $v = v_0 s(v_\infty, v) + v_\infty s(v_0, v) = (v_0 + v_\infty\mathfrak{p})s(v_\infty, v)$. Hence, $s(v, v) = 0$ implies $\mathfrak{p} + \bar{\mathfrak{p}} = 0$, such that the stereographic projection takes values in the imaginary quaternions. A different choice of homogeneous coordinates $\tilde{v}_0 = v_0 a$ now determines $\tilde{v}_\infty = v_\infty\bar{a}^{-1}$ — the effect on the stereographic projection being a stretch-rotation $\tilde{\mathfrak{p}} = a\mathfrak{p}\bar{a}$ in $I\!\!R^3 \cong \mathrm{Im}I\!\!H$.

3. Transformations of discrete isothermic nets

Now, let $f : \mathbb{Z}^2 \to I\!\!HP^1$ denote a discrete map, φ the corresponding map into $(I\!\!HP^1)^\perp$, i.e. $\varphi \circ f = 0$, and let $\mathfrak{f} : \mathbb{Z}^2 \to I\!\!H$ denote its stereographic projection with respect to (v_0, v_∞), or its pseudo dual basis (ν_0, ν_∞) respectively. To

simplify formulas, we will assume $\varphi v_\infty = \nu_\infty f \equiv 1$ wherever a choice of homogeneous coordinates for two points $0, \infty$ plays role. Then (4) yields

$$
\begin{array}{rclclcl}
-\varphi_{m+1,n} f_{m,n} & = & \varphi_{m,n} f_{m+1,n} & = & \mathfrak{f}_{m+1,n} - \mathfrak{f}_{m,n} & =: & (\partial_1 \mathfrak{f})_{m,n}, \\
-\varphi_{m,n+1} f_{m,n} & = & \varphi_{m,n} f_{m,n+1} & = & \mathfrak{f}_{m,n+1} - \mathfrak{f}_{m,n} & =: & (\partial_2 \mathfrak{f})_{m,n}.
\end{array} \tag{5}
$$

Thus, a minimal regularity assumption on f is suggested:

Definition (Discrete net). *A discrete map $\varphi \simeq f : \mathbb{Z}^2 \to \mathbb{H}P^1 \cong (\mathbb{H}P^1)^\perp$, or $\mathfrak{f} : \mathbb{Z}^2 \to \mathbb{H}$ respectively, will be called a discrete net if $(\partial_1 \mathfrak{f})_{m,n} \neq 0 \neq (\partial_2 \mathfrak{f})_{m,n}$ and $(\partial_1 \mathfrak{f})_{m,n} \nparallel (\partial_2 \mathfrak{f})_{m,n}$ are non-parallel for all $(m,n) \in \mathbb{Z}^2$.*

For the rest of the paper, we will adopt the above notation conventions; in case of discrete nets $f : \mathbb{Z}^2 \to S^3 \cong \mathbb{R}s \subset \mathbb{R}_1^6$ into the conformal 3-sphere, we will assume $s(v_0, v_\infty) = 1$ and $\varphi = s(f, .)$.

As an analog of an arbitrary (not necessarily conformal) curvature line parametrization of a smooth isothermic surface we use the following "wide definition" (cf.[6]) for discrete isothermic nets:

Definition (Isothermic net). *A discrete net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ is called isothermic if all cross ratios $q_{m,n} = [f_{m,n}, f_{m+1,n}, f_{m+1,n+1}, f_{m,n+1}] = \frac{a_m}{b_n}$ where the (real valued) functions $a, b : \mathbb{Z} \to \mathbb{R}$ depend on m resp. n only.*

This way, isothermic nets are a special case of discrete principal nets [6]:

Definition (Principal net). *A discrete net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ is called principal net if all elementary quadrilaterals are concircular, i.e. if $q : \mathbb{Z}^2 \to \mathbb{R}$.*

3.1 The difference equations

Smooth isothermic surfaces $f : M^2 \to S^3$ can be characterized as the only "Möbius deformable" surfaces, i.e. by the existence of a family of non constant maps $T^\lambda : M^2 \to M\ddot{o}b(S^3)$ such that $f^\lambda := T^\lambda f : M^2 \to S^3$ induce the same curvature lines and conformal metric[3] (cf.[10],[17]). Classically, this deformation is known as the "T-transformation" [2]; we first give a discrete version of the corresponding system of differential equations in the quaternionic setting [9],[15]:

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[3] The term "conformal metric" is used for the induced metric of the central sphere congruence (conformal Gauss map) of the surface; the "conformal Hopf differential" of a surface encodes both informations, thus it determines an immersion f uniquely up to Möbius transformation unless f parametrizes an isothermic surface (cf.[18]).

Lemma (T-transformation). *Let $f : \mathbb{Z}^2 \to I\!\!HP^1$, $f \simeq \varphi : \mathbb{Z}^2 \to (I\!\!HP^1)^{\perp}$, be a discrete net, and $\nu_{\infty}f = \varphi v_{\infty} \equiv 1$ after some choice of homogeneous coordinates $v_0, v_{\infty} \in I\!\!H^2$ for two points $0, \infty \in I\!\!HP^1$; we define*

$$U_{m,n} \quad := \quad f_{m,n}u_{m,n}\varphi_{m+1,n} \qquad \text{and} \qquad V_{m,n} \quad := \quad f_{m,n}v_{m,n}\varphi_{m,n+1} \quad (6)$$

with some functions $u, v : \mathbb{Z}^2 \to I\!\!H$. Then, the system

$$T^{\lambda}_{m+1,n} \quad = \quad T^{\lambda}_{m,n}(1 + \lambda U_{m,n}) \quad , \quad T^{\lambda}_{m,n+1} \quad = \quad T^{\lambda}_{m,n}(1 + \lambda V_{m,n}) \qquad (7)$$

is solvable for all $\lambda \in I\!\!R$ if and only if the following system is:

$$\begin{array}{rcl} F^{\lambda}_{m+1,n} & = & F^{\lambda}_{m,n}(1 + v_{\infty}(\partial_1\mathfrak{f})_{m,n}\nu_{\infty} + \lambda v_0 u_{m,n}\nu_0), \\ F^{\lambda}_{m,n+1} & = & F^{\lambda}_{m,n}(1 + v_{\infty}(\partial_2\mathfrak{f})_{m,n}\nu_{\infty} + \lambda v_0 v_{m,n}\nu_0). \end{array} \qquad (8)$$

Proof. Since $\varphi v_{\infty} = \nu_{\infty}f \equiv 1$, we have $\nu_0 f = \mathfrak{f} = -\varphi v_0$ as $\varphi f \equiv 0$. Thus, with the ansatz $F^0 := id + v_{\infty}\mathfrak{f}\nu_{\infty}$ for the "Euclidean frame" of f, a straightforward calculation yields

$$\begin{array}{rcl} (F^0_{m,n})^{-1}(1 + \lambda U_{m,n}) F^0_{m+1,n} & = & (id + v_{\infty}(\partial_1\mathfrak{f})_{m,n}\nu_{\infty} + \lambda v_0 u_{m,n}\nu_0), \\ (F^0_{m,n})^{-1}(1 + \lambda V_{m,n}) F^0_{m,n+1} & = & (id + v_{\infty}(\partial_2\mathfrak{f})_{m,n}\nu_{\infty} + \lambda v_0 v_{m,n}\nu_0), \end{array}$$

i.e. the gauge equivalence of the systems (7) and (8). ◁

This lemma gives the relation between the discrete "curved flat" system (8) and the system (7) modelling the conformal deformability of (isothermic) nets: using (5) and denoting $a = u \cdot (\partial_1\mathfrak{f})$ and $b = v \cdot (\partial_2\mathfrak{f})$, we have

$$\begin{array}{rcl} (1 + \lambda U_{m,n})f_{m,n} & = & f_{m,n}(1 - \lambda a_{m,n}), \\ (1 + \lambda U_{m,n})f_{m+1,n} & = & f_{m+1,n}; \\ (1 + \lambda V_{m,n})f_{m,n} & = & f_{m,n}(1 - \lambda b_{m,n}), \\ (1 + \lambda V_{m,n})f_{m,n+1} & = & f_{m,n+1}. \end{array} \qquad (9)$$

Thus, the T^{λ} (if they exist) map a "vertex star" consisting of a center vertex $f_{m,n}$ and its four neighbours $f_{m\pm1,n}$ and $f_{m,n\pm1}$ to

$$\begin{array}{rcl} T^{\lambda}_{m,n}f_{m,n} & = & T^{\lambda}_{m,n}f_{m,n} \\ T^{\lambda}_{m-1,n}f_{m-1,n} & = & T^{\lambda}_{m,n}f_{m-1,n}(1 - \lambda a_{m-1,n})^{-1} \\ T^{\lambda}_{m+1,n}f_{m+1,n} & = & T^{\lambda}_{m,n}f_{m+1,n} \\ T^{\lambda}_{m,n-1}f_{m,n-1} & = & T^{\lambda}_{m,n}f_{m,n-1}(1 - \lambda b_{m,n-1})^{-1} \\ T^{\lambda}_{m,n+1}f_{m,n+1} & = & T^{\lambda}_{m,n}f_{m,n+1} \end{array} \qquad (10)$$

i.e. any vertex star and its image under the T^{λ} are Möbius equivalent. In this sense, $\lambda \to T^{\lambda}_{m,n}f_{m,n}$ can be considered a "Möbius-deformation" of f:

Definition (Möbius-deformation). *Let $f : \mathbb{Z}^2 \to I\!\!HP^1$ be a discrete net; a family $(T^{\lambda})_{\lambda}$, $T^{\lambda} : \mathbb{Z}^2 \to Gl(2, I\!\!H)$, will be called a Möbius-deformation for f if any corresponding vertex stars of $f = T^0 f$ and $T^{\lambda}f$ are Möbius equivalent.*

The integrability conditions for (7) directly lead to a discrete analog for the "general C-transform" for an isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}$ [9]:

Lemma (C-transformation). *The system (7) is integrable for all $\lambda \in \mathbb{R}$ if and only if $U = \partial_1 \mathfrak{F}^*$ and $V = \partial_2 \mathfrak{F}^*$ with some $\mathfrak{F}^* : \mathbb{Z}^2 \to \mathfrak{gl}(2, \mathbb{H})$:*

$$\mathfrak{F}^*_{m+1,n} = \mathfrak{F}^*_{m,n} + U_{m,n} \quad , \quad \mathfrak{F}^*_{m,n+1} = \mathfrak{F}^*_{m,n} + V_{m,n} \quad . \tag{11}$$

Proof. The discrete version of the Maurer-Cartan equations for the system (7) reads $(1 + \lambda U_{m,n})(1 + \lambda V_{m+1,n}) = (1 + \lambda V_{m,n})(1 + \lambda U_{m,n+1})$; as $\varphi \circ f \equiv 0$ this simplifies to $\lambda(U_{m,n} + V_{m+1,n}) = \lambda(V_{m,n} + U_{m,n+1})$, which is the integrability condition for (11) when $\lambda = 1$. ◁

3.2 The C-transformation

Certain projections then give (as we will see a bit later) the well known [6] discrete analog of the Christoffel transformation in Euclidean space (cf.[9]):

Lemma. *Let $v_0, v_\infty \in \mathbb{H}^2$ be homogeneous coordinates of $0, \infty \in \mathbb{HP}^1$. Then, (11) is integrable if and only if there is a discrete map $\mathfrak{f}^* = \nu_\infty \mathfrak{F}^* v_\infty : \mathbb{Z}^2 \to \mathbb{H}$ such that $u = \partial_1 \mathfrak{f}^*$ and $v = \partial_2 \mathfrak{f}^*$, and*

$$\begin{aligned}
(\partial_1 \mathfrak{f}^*)_{m,n}(\partial_2 \mathfrak{f})_{m+1,n} &= (\partial_2 \mathfrak{f}^*)_{m,n}(\partial_1 \mathfrak{f})_{m,n+1}, \\
(\partial_1 \mathfrak{f})_{m,n}(\partial_2 \mathfrak{f}^*)_{m+1,n} &= (\partial_2 \mathfrak{f})_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n+1}.
\end{aligned} \tag{12}$$

Moreover, with $\mathfrak{d}_{m,n} := (\partial_1 \mathfrak{f})_{m,n} - (\partial_2 \mathfrak{f})_{m,n} = (\partial_1 \mathfrak{f})_{m,n+1} - (\partial_2 \mathfrak{f})_{m+1,n}$,

$$\begin{aligned}
\mathfrak{d}_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n+1}(\partial_1 \mathfrak{f})_{m,n+1} &= (\partial_1 \mathfrak{f})_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n}\mathfrak{d}_{m,n}, \\
\mathfrak{d}_{m,n}(\partial_2 \mathfrak{f}^*)_{m+1,n}(\partial_2 \mathfrak{f})_{m+1,n} &= (\partial_2 \mathfrak{f})_{m,n}(\partial_2 \mathfrak{f}^*)_{m,n}\mathfrak{d}_{m,n}; \\
\mathfrak{d}_{m,n}(\partial_2 \mathfrak{f}^*)_{m+1,n}(\partial_1 \mathfrak{f})_{m,n+1} &= (\partial_2 \mathfrak{f})_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n}\mathfrak{d}_{m,n}, \\
\mathfrak{d}_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n+1}(\partial_2 \mathfrak{f})_{m+1,n} &= (\partial_1 \mathfrak{f})_{m,n}(\partial_2 \mathfrak{f}^*)_{m,n}\mathfrak{d}_{m,n}.
\end{aligned} \tag{13}$$

Proof. A straightforward computation shows that the integrability condition for (8) is equivalent to $u_{m,n} + v_{m+1,n} = v_{m,n} + u_{m,n+1}$, i.e. $u = \partial_1 \mathfrak{f}^*$ and $v = \partial_2 \mathfrak{f}^*$ for some discrete map $\mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$, together with (12). Since $\nu_\infty f = 1 = \varphi v_\infty$, $\nu_\infty U v_\infty = u$ and $\nu_\infty V v_\infty = v$ and hence $\mathfrak{f}^* = \nu_\infty \mathfrak{F}^* v_\infty$. Moreover, using (12), each of the equations (13) becomes the integrability condition for $\mathfrak{f}^*$. ◁

Note that (12) yields a discrete version of the equations $d\mathfrak{f} \wedge d\mathfrak{f}^* = 0 = d\mathfrak{f}^* \wedge d\mathfrak{f}$ characterizing the Christoffel transform $\mathfrak{f}^* : M^2 \to \mathbb{H}$ in the smooth case [14].

Assuming that $\mathfrak{f}^*$ is a discrete *net*, another useful formula for the cross ratios of elementary quadrilaterals of f can be obtained as a direct consequence of (12):

Lemma. $[f_{m,n}, f_{m+1,n}, f_{m+1,n+1}, f_{m,n+1}] = \operatorname{Re} q_{m,n} + i|\operatorname{Im} q_{m,n}|$ *where*

$$q_{m,n} = [(\partial_1 \mathfrak{f}^*)(\partial_1 \mathfrak{f})]_{m,n}[(\partial_2 \mathfrak{f}^*)(\partial_2 \mathfrak{f})]_{m,n}^{-1} \tag{14}$$

if $\mathfrak{f}^*$ is a discrete net, satisfying (12) with some stereographic projection $\mathfrak{f}$ of f.

If the functions $a := (\partial_1 \mathfrak{f}^*)(\partial_1 \mathfrak{f}), b := (\partial_2 \mathfrak{f}^*)(\partial_2 \mathfrak{f}) : \mathbb{Z}^2 \to \mathbb{R}$ are real valued, then the first two equations of (13) show (the real part of a quaternionic product is symmetric) that $a = a_m$ and $b = b_n$ depend on m resp. n only. Thus, (14) implies that $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ is a discrete isothermic net; and, $\mathfrak{f}^*$ is a Christoffel transform [6] of the stereographic projection $\mathfrak{f} : \mathbb{Z}^2 \to \mathbb{H}$ of f — that, for symmetry reasons, is an isothermic net with $q^* = q$, too:

Definition (C-transformation). *Let $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ be a discrete isothermic net, $q_{m,n} = \frac{a_m}{b_n}$, $\mathfrak{f} = (\nu_0 f)(\nu_\infty f)^{-1} : \mathbb{Z}^2 \to \mathbb{H}$ a stereographic projection of f. Then, the discrete isothermic net $\mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$ given by $(\partial_1 \mathfrak{f}^*) = a\,(\partial_1 \mathfrak{f})^{-1}$ and $(\partial_2 \mathfrak{f}^*) = b\,(\partial_2 \mathfrak{f})^{-1}$ is called a C-transform (Christoffel transform) of $\mathfrak{f}$.*

Note, that the C-transform $\mathfrak{f}^*$ of $\mathfrak{f}$ is unique up to scale and translation in $\mathbb{H}$.

3.3 The G-transformation

Obviously, the C-transformation depends on the Euclidean structure of the ambient space; on the other hand, the notion of an isothermic net is a conformal notion. Using this interplay of geometries, a new transformation can be introduced — as in the smooth case where this transformation generalizes the classical Goursat transformation for minimal surfaces [15]:

Definition (G-transformation). *Two isotermic nets $\mathfrak{f}, \tilde{\mathfrak{f}} : Z^2 \to \mathbb{H}$ are said to be G-transforms (Goursat transforms) of each other if their C-transforms are stereographic projections of the same isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$.*

As the C-transformation is involutive, $\mathfrak{f}^{**} \simeq \mathfrak{f}$ up to scaling and translation, the G-transformations of an isothermic net $\mathfrak{f} : \mathbb{Z}^2 \to \mathbb{H}$ form a group. Generically, there is a 4-parameter family of G-transforms of a given isothermic net that are not congruent to the original net: if $\mathfrak{f}^*$ and $\tilde{\mathfrak{f}}^*$ denote C-transforms of different stereographic projections $\mathfrak{f} = (\nu_0 f)$ and $\tilde{\mathfrak{f}} = (\tilde{\nu}_0 f)(\tilde{\nu}_\infty f)^{-1}$ of an isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ (as before, without loss of generality $\nu_\infty f \equiv 1$), then (5) yields

$$
\begin{aligned}
(\partial_1 \tilde{\mathfrak{f}})_{m,n} &= (\varphi_{m+1,n} \tilde{v}_\infty)^{-1}(\partial_1 \mathfrak{f})_{m,n}(\tilde{\nu}_\infty f_{m,n})^{-1} \\
\Leftrightarrow \quad (\partial_1 \tilde{\mathfrak{f}}^*)_{m,n} &= (\tilde{\nu}_\infty f_{m,n})(\partial_1 \mathfrak{f}^*)_{m,n}(\varphi_{m+1,n} \tilde{v}_\infty), \\
(\partial_2 \tilde{\mathfrak{f}})_{m,n} &= (\varphi_{m,n+1} \tilde{v}_\infty)^{-1}(\partial_2 \mathfrak{f})_{m,n}(\tilde{\nu}_\infty f_{m,n})^{-1} \\
\Leftrightarrow \quad (\partial_2 \tilde{\mathfrak{f}}^*)_{m,n} &= (\tilde{\nu}_\infty f_{m,n})(\partial_2 \mathfrak{f}^*)_{m,n}(\varphi_{m,n+1} \tilde{v}_\infty).
\end{aligned}
\tag{15}
$$

Thus, $\mathfrak{f}^*$ and $\tilde{\mathfrak{f}}^*$ will generically be non-congruent if $\tilde{v}_\infty \mathbb{H} \neq v_\infty \mathbb{H}$ are different points in $\mathbb{H}P^1$.

Clearly, any "Christoffel pair" $\mathfrak{f}, \mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$ in $\mathbb{H}$, consisting of a discrete isothermic net $\mathfrak{f}$ and a Christoffel transform $\mathfrak{f}^*$ of $\mathfrak{f}$, satisfies (12). Thus, (11) is integrable for $f = v_0 + v_\infty \mathfrak{f}$ and $\varphi = \nu_0 - \mathfrak{f}\nu_\infty$, $u = \partial_1 \mathfrak{f}^*$ and $v = \partial_2 \mathfrak{f}^*$: the "general C-transform" $\mathfrak{F}^* : \mathbb{Z}^2 \to \mathfrak{gl}(2, \mathbb{H})$ of an isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ encodes the C-transforms of all possible stereographic projections of f — therefore, any two

"projections" $\nu_\infty \mathfrak{F}^* v_\infty$ and $\tilde{\nu}_\infty \mathfrak{F}^* \tilde{v}_\infty$ into Euclidean space are G-transforms of each other. Choosing another stereographic projection $\tilde{\mathfrak{f}} = (\tilde{\nu}_0 f)(\tilde{\nu}_\infty f)^{-1}$ of f from the beginning, the C-transform $\mathfrak{f}^*$ changes by a G-transformation — (15) then shows that U and V in (6),

$$\begin{array}{rclcl} \tilde{U}_{m,n} & = & [f(\tilde{\nu}_\infty f)^{-1}]_{m,n}(\partial_1 \tilde{\mathfrak{f}}^*)_{m,n}[(\varphi \tilde{v}_\infty)^{-1}\varphi]_{m+1,n} & = & U_{m,n} \\ \tilde{V}_{m,n} & = & [f(\tilde{\nu}_\infty f)^{-1}]_{m,n}(\partial_2 \tilde{\mathfrak{f}}^*)_{m,n}[(\varphi \tilde{v}_\infty)^{-1}\varphi]_{m,n+1} & = & V_{m,n}, \end{array}$$

do not depend on the choice of stereographic projection $\mathfrak{f}$. Hence, $\mathfrak{F}^*$ itself does not depend on the choice of stereographic projection — and, "contains" the G-equivalence class of all possible C-transforms of an isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ as "off-diagonal elements" $\mathfrak{f}^* = \nu_\infty \mathfrak{F}^* v_\infty$:

Definition (C-transformation). *Given an isothermic net $f : \mathbb{Z}^2 \to \mathbb{H}P^1$, a solution $\mathfrak{F}^* : \mathbb{Z}^2 \to \mathfrak{gl}(2, \mathbb{H})$ of (11) is called the* general C-transform of f.

3.4 The T-transformation

With the same argument, a solution T^λ of (7) does not depend on the choice of stereographic projection. Consequently, the following definition makes sense:

Definition (T-transformation). *Let $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ be an isothermic net, $\mathfrak{f}^*$ a C-transform of any stereographic projection $\mathfrak{f} = (\nu_0 f)(\nu_\infty f)^{-1}$ of f; moreover, let $T^\lambda : \mathbb{Z}^2 \to Gl(2, \mathbb{H})$ be a solution of (7) with $u = \partial_1 \mathfrak{f}^*$ and $v = \partial_2 \mathfrak{f}^*$:*

$$\begin{array}{rcl} T^\lambda_{m+1,n} & = & T^\lambda_{m,n}(1 + \lambda[f(\nu_\infty f)^{-1}]_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n}[(\varphi v_\infty)^{-1}\varphi]_{m+1,n}), \\ T^\lambda_{m,n+1} & = & T^\lambda_{m,n}(1 + \lambda[f(\nu_\infty f)^{-1}]_{m,n}(\partial_2 \mathfrak{f}^*)_{m,n}[(\varphi v_\infty)^{-1}\varphi]_{m,n+1}). \end{array} \qquad \text{(T)}$$

Then, $f^\lambda := T^\lambda f : \mathbb{Z}^2 \to \mathbb{H}P^1$ is called a T-transform of f.

As the C-transform $\mathfrak{f}^*$ is unique up to scale and translation, the family $(f^\lambda)_{\lambda \in \mathbb{R}}$ of T-transforms of f is unique up to Möbius transformations; fixing a factorization of the cross ratios, $q_{m,n} = \frac{a_m}{b_n}$, each f^λ is unique up to a Möbius transformation.

Generically, a T-transform f^λ of an isothermic net f is a discrete net:

Lemma. *Let $f : \mathbb{Z}^2 \to \mathbb{H}P^1$ be an isothermic net, $q_{m,n} = \frac{a_m}{b_n}$. Choosing an invertible initial condition, $T^\lambda_{0,0} \in Gl(2, \mathbb{H})$, a solution T^λ of (T) stays invertible if and only if $\lambda a_m \neq 1 \neq \lambda b_n$.*

Proof. By (9), $(1 + \lambda U), (1 + \lambda V) : \mathbb{Z}^2 \to Gl(2, \mathbb{H})$ iff $\lambda a_m \neq 1 \neq \lambda b_n$ since the regularity condition on f (f is a discrete *net*) implies that $f_{m,n}, f_{m+1,n}$ and $f_{m,n}, f_{m,n+1}$ are (quaternionic) linearly independent.   ◁

Moreover, if f^λ is a discrete net, then it is isothermic:

Lemma. *Let f^λ be a non-degenerate T-transform of an isothermic net f with $q_{m,n} = \frac{a_m}{b_n}$; then f^λ is isothermic with $q^\lambda_{m,n} = q_{m,n}\frac{1-\lambda b_n}{1-\lambda a_m} = \frac{a_m}{1-\lambda a_m} : \frac{b_n}{1-\lambda b_n}$.*

Proof. $\varphi^\lambda = \varphi(T^\lambda)^{-1}$ as $f^\lambda = T^\lambda f$. Consequently, using (9),

$$
\begin{aligned}
\varphi^\lambda_{m,n} f^\lambda_{m+1,n} &= & \varphi_{m,n}(1 + \lambda U_{m,n})f_{m+1,n} &= & \varphi_{m,n}f_{m+1,n}, \\
\varphi^\lambda_{m+1,n+1} f^\lambda_{m+1,n} &= & \varphi_{m+1,n+1}(1 + \lambda V_{m+1,n})^{-1}f_{m+1,n} &= & \tfrac{\varphi_{m+1,n+1}f_{m+1,n}}{1-\lambda b_n}, \\
\varphi^\lambda_{m+1,n+1} f^\lambda_{m,n+1} &= & \varphi_{m+1,n+1}(1 + \lambda U_{m,n+1})^{-1}f_{m,n+1} &= & \tfrac{\varphi_{m+1,n+1}f_{m,n+1}}{1-\lambda a_m}, \\
\varphi^\lambda_{m,n} f^\lambda_{m,n+1} &= & \varphi_{m,n}(1 + \lambda V_{m,n})f_{m,n+1} &= & \varphi_{m,n}f_{m,n+1}.
\end{aligned}
$$

Thus, (2) yields the result $q^\lambda_{m,n} = q_{m,n}\frac{1-\lambda b_n}{1-\lambda a_m}$.   ◁

This shows that the "regular" T-transformations act on the space of (Möbius equivalence classes of) isothermic nets. The next lemma shows that (for "regular values" of λ) the T-transformations act like a "1-parameter group" on the (1-dimensional) orbit $(f^\lambda)_\lambda$ of a fixed isothermic net:

Theorem. *Let $f = f^0 : \mathbb{Z}^2 \to \mathbb{HP}^1$ be an isothermic net and $(T^\lambda)_{\lambda \in \mathbb{R}}$ its family of T-transformations, $T^\lambda_{0,0} = id$ for all $\lambda \in \mathbb{R}$. If $T^{\lambda_1}, T^{\lambda_2} : \mathbb{Z}^2 \to Gl(2, \mathbb{H})$ are non-degenerate, then $T^{\lambda_1 + \lambda_2} = T^{\lambda_2}T^{\lambda_1}$. In particular, $(T^\lambda)^{-1} = T^{-\lambda}$.*

Proof. Using (9), $T^{\lambda_1}_{m+1,n} = T^{\lambda_1}_{m,n}(1 + \lambda_1 U_{m,n})$ and $T^{\lambda_1}_{m,n+1} = T^{\lambda_1}_{m,n}(1 + \lambda_1 V_{m,n})$,

$$
\begin{aligned}
(1 + \lambda_2 U^{\lambda_1}_{m,n})T^{\lambda_1}_{m,n}(1 + \lambda_1 U_{m,n})f_{m,n} &= & f^{\lambda_1}_{m,n}(1 - (\lambda_1 + \lambda_2)a_m), \\
(1 + \lambda_2 U^{\lambda_1}_{m,n})T^{\lambda_1}_{m,n}(1 + \lambda_1 U_{m,n})f_{m+1,n} &= & f^{\lambda_1}_{m+1,n}, \\
(1 + \lambda_2 V^{\lambda_1}_{m,n})T^{\lambda_1}_{m,n}(1 + \lambda_1 V_{m,n})f_{m,n} &= & f^{\lambda_1}_{m,n}(1 - (\lambda_1 + \lambda_2)b_n), \\
(1 + \lambda_2 V^{\lambda_1}_{m,n})T^{\lambda_1}_{m,n}(1 + \lambda_1 V_{m,n})f_{m,n+1} &= & f^{\lambda_1}_{m,n+1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(T^{\lambda_1})^{-1}(1 + \lambda_2 U^{\lambda_1})T^{\lambda_1}(1 + \lambda_1 U) &= & 1 + (\lambda_1 + \lambda_2)U, \\
(T^{\lambda_1})^{-1}(1 + \lambda_2 V^{\lambda_1})T^{\lambda_1}(1 + \lambda_1 V) &= & 1 + (\lambda_1 + \lambda_2)V.
\end{aligned}
$$

Therefore, $T^{\lambda_2}T^{\lambda_1}$ and $T^{\lambda_1 + \lambda_2}$ satisfy the same difference equations; since they coincide at $(m,n) = (0,0)$, $T^{\lambda_2}T^{\lambda_1} = T^{\lambda_1 + \lambda_2}$. In particular, $(T^\lambda)^{-1} = T^{-\lambda}$.   ◁

Finally, we obtain some type of discrete version of the non-rigidity result for isothermic surfaces in 3-dimensional Möbius geometry:

Lemma. *Let $f : \mathbb{Z}^2 \to S^3$ be a discrete net in a 3-sphere $S^3 \subset \mathbb{HP}^1$; then, a solution $T^\lambda : \mathbb{Z}^2 \to M\ddot{o}b(S^3)$ of (7) takes — up to a fixed Möbius transformation — values in the Möbius group of S^3 if and only if f is isothermic.*

Proof. Choose homogeneous coordinates $s \in I\!\!R_1^6$ for S^3, and let $a := (\partial_1 f^*)(\partial_1 f)$ and $b := (\partial_2 f^*)(\partial_2 f)$; then, (9) yields

$$
\begin{aligned}
((1 + \lambda U_{m,n})^{-1} \cdot s)(f_{m,n}, f_{m,n}) &= ((1 + \lambda U_{m,n})^{-1} \cdot s)(f_{m+1,n}, f_{m+1,n}) \\
&= 0 \qquad\qquad \text{and} \\
((1 + \lambda U_{m,n})^{-1} \cdot s)(f_{m+1,n}, f_{m,n}) &= s(f_{m+1,n}, f_{m,n})(1 - \lambda a_{m,n})^{-1}; \\
((1 + \lambda V_{m,n})^{-1} \cdot s)(f_{m,n}, f_{m,n}) &= ((1 + \lambda V_{m,n})^{-1} \cdot s)(f_{m,n+1}, f_{m,n+1}) \\
&= 0 \qquad\qquad \text{and} \\
((1 + \lambda V_{m,n})^{-1} \cdot s)(f_{m,n+1}, f_{m,n}) &= s(f_{m,n+1}, f_{m,n})(1 - \lambda b_{m,n})^{-1}.
\end{aligned}
$$

Thus, $(1 + \lambda U), (1 + \lambda V) : Z^2 \to M\ddot{o}b(S^3)$ preserve the sphere $S^3 \cong I\!\!R s$ if and only if $a, b : \mathbb{Z}^2 \to I\!\!R$ are real valued. But, we already deduced it as a consequence of (13), that this is equivalent to f being isothermic with $q_{m,n} = \frac{a_m}{b_n}$ where the functions $a, b : \mathbb{Z} \to I\!\!R$ depend on one variable only. $\triangleleft$
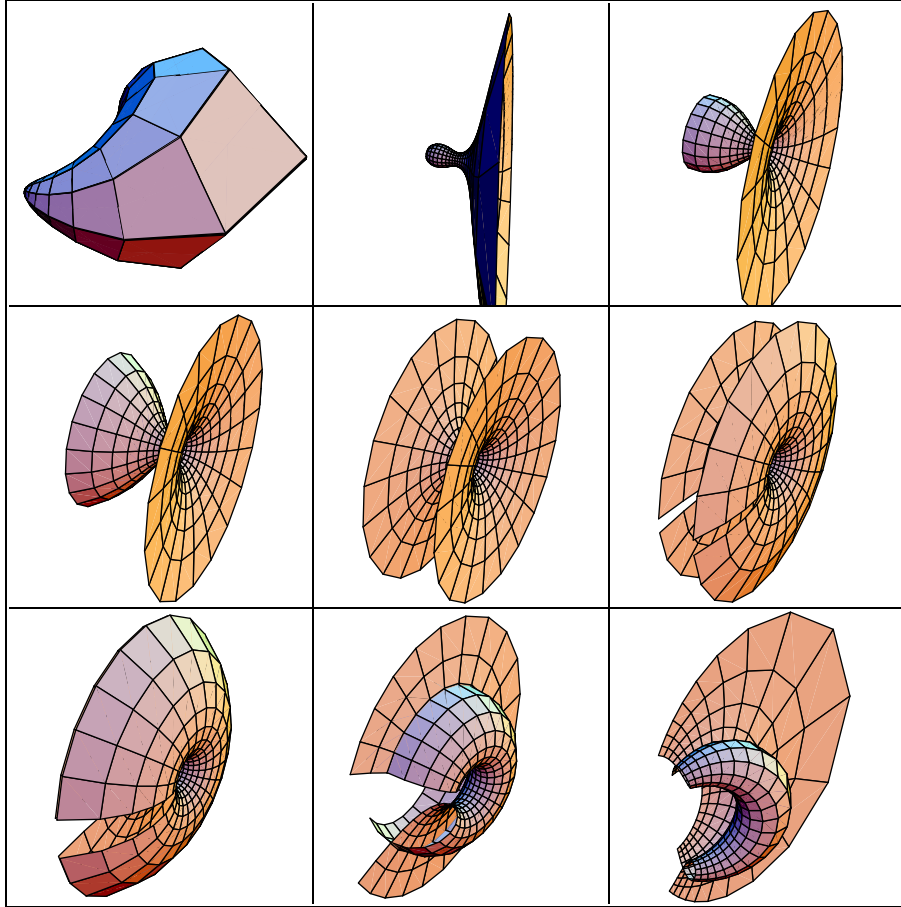

Fig. 1. T-transforms of a discrete Catenoid with Gauss map $\mathfrak{g}_{m,n} = e^{2\pi \frac{m+in}{20}}$

The following consequence will prove crucial for our approach to a discrete analog for surfaces of constant mean curvature 1 in hyperbolic space:

Corollary. *If $f : \mathbb{Z}^2 \to S^3$ is an isothermic net, then (up to a Möbius transformation) its T-transforms $f^\lambda : \mathbb{Z}^2 \to S^3$ take values in S^3, too.*
If $f : \mathbb{Z}^2 \to S^2$, then $f^\lambda : \mathbb{Z}^2 \to S^2$.

3.5 The D-transformation

Finally, we formulate the discrete analog of the Darboux transformation in the conformal setting:

Definition (D-transformation). *A discrete net $\hat{f} : \mathbb{Z}^2 \to I\!\!H P^1$ is called a D-transform (Darboux transform) of an isothermic net $f : \mathbb{Z}^2 \to I\!\!H P^1$, $\hat{f} = D_\lambda f$, if $T^\lambda \hat{f} = const \in I\!\!H P^1$ is a fixed point.*

Obviously, for fixed λ and fixed "initial point" $\hat{f}_{0,0} \in I\!\!H P^1$ there is a unique Darboux transform $(T^\lambda)^{-1}(T_{0,0}^\lambda \hat{f}_{0,0}) : \mathbb{Z}^2 \to I\!\!H P^1$ of an isothermic net f. If f is an isothermic net on a 3-sphere S^3, then $T^\lambda : Z^2 \to M\ddot{o}b(S^3)$ so that any D-transform with initial point $\hat{f}_{0,0} \in S^3$ stays on S^3. Thus, a fixed isothermic net f allows a ∞^{1+4} D-transforms in $I\!\!H P^1$, or ∞^{1+3} D-transforms in S^3.

Actually, the above definition for the D-transformation agrees with the one given in [16], where we used a discrete version of the Riccati equation characterizing smooth Darboux transforms in Euclidean 4-space:

Lemma. *A discrete net $\hat{f} : \mathbb{Z}^2 \to I\!\!H P^1$ is a Darboux transform of an isothermic net $f : \mathbb{Z}^2 \to I\!\!H P^1$ if and only if their stereographic projections $\mathfrak{f}, \hat{\mathfrak{f}} : Z^2 \to I\!\!H$ satisfy the Riccati type system*

$$
\begin{array}{rcl}
(\partial_1 \hat{\mathfrak{f}})_{m,n} & = & \lambda(\hat{\mathfrak{f}} - \mathfrak{f})_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n}(\hat{\mathfrak{f}} - \mathfrak{f})_{m+1,n}, \\
(\partial_2 \hat{\mathfrak{f}})_{m,n} & = & \lambda(\hat{\mathfrak{f}} - \mathfrak{f})_{m,n}(\partial_2 \mathfrak{f}^*)_{m,n}(\hat{\mathfrak{f}} - \mathfrak{f})_{m,n+1}.
\end{array}
\tag{16}
$$

Proof. $\hat{f} : \mathbb{Z}^2 \to I\!\!H P^1$ being a D-transform of an isothermic net $f : \mathbb{Z}^2 \to I\!\!H P^1$ is equivalent to $(1 + \lambda U_{m,n})\hat{f}_{m+1,n} \parallel \hat{f}_{m,n} \parallel (1 + \lambda V_{m,n})\hat{f}_{m,n+1}$ for all m, n. As an equivalent formulation, we obtain

$$
\begin{array}{rcl}
0 & = & \hat{\varphi}_{m,n}(1 + \lambda U_{m,n})\hat{f}_{m+1,n} \\
  & = & (\partial_1 \hat{\mathfrak{f}})_{m,n} + \lambda(\mathfrak{f} - \hat{\mathfrak{f}})_{m,n}(\partial_1 \mathfrak{f}^*)_{m,n}(\hat{\mathfrak{f}} - \mathfrak{f})_{m+1,n} \\
0 & = & \hat{\varphi}_{m,n}(1 + \lambda V_{m,n})\hat{f}_{m,n+1} \\
  & = & (\partial_2 \hat{\mathfrak{f}})_{m,n} + \lambda(\mathfrak{f} - \hat{\mathfrak{f}})_{m,n}(\partial_2 \mathfrak{f}^*)_{m,n}(\hat{\mathfrak{f}} - \mathfrak{f})_{m,n+1}
\end{array}
$$

from (5) — remember that we agreed on having $\nu_\infty \hat{f} \equiv 1 \equiv \hat{\varphi} v_\infty$. $\triangleleft$

This lemma allows to carry over the results given in [16] to the more general setting[4] of isothermic nets $f : \mathbb{Z}^2 \to I\!\!H P^1$ "in the wide sense". As in [16],

---

[4] In [16], we restricted to the special case $a_m = +1$, $b_n = -1$ of the discrete analog of *conformal* curvature line coordinates of isothermic surfaces in $I\!\!H \cong I\!\!R^4$.

(16) can be reformulated to obtain cross ratio conditions for the quadrilaterals formed by corresponding edges on f and $\hat{f}$:

$$
\begin{aligned}
\lambda a_m &= [f_{m,n}, f_{m+1,n}, \hat{f}_{m+1,n}, \hat{f}_{m,n}], \\
\lambda b_n &= [f_{m,n}, f_{m,n+1}, \hat{f}_{m,n+1}, \hat{f}_{m,n}].
\end{aligned} \tag{17}
$$

With this observation, several key results on the Darboux transformation can be reduced (cf.[16]) to the

Lemma (Hexahedron lemma). *Let* $p_1, \ldots, p_4 \in I\!H P^1$ *be four concircular points, i.e.* $[p_1, p_2, p_3, p_4] =: \mu \in I\!R$, *and* $\lambda \in I\!R$. *Then, to any* $q_1 \in I\!H P^1$ *there exist unique points* $q_2, q_3, q_4 \in I\!H P^1$ *such that*

$$
\begin{aligned}
{[q_1, q_2, q_3, q_4]} &= [p_1, p_2, p_3, p_4] &= \mu \\
{[q_1, q_2, p_2, p_1]} &= [q_3, q_4, p_4, p_3] &= \mu\lambda \\
{[q_2, q_3, p_3, p_2]} &= [q_4, q_1, p_1, p_4] &= \lambda
\end{aligned}
$$

Moreover, the eight points p_i, q_i *lie on a 2-sphere.*

As a direct consequence (cf.[16],[4]),

Theorem. *A D-transform* $\hat{f} : Z^2 \to I\!H P^1$ *of an isothermic net* $f : \mathbb{Z}^2 \to I\!H P^1$ *is an isothermic net with the same cross ratios* $\hat{q}_{m,n} = q_{m,n}$; *moreover, f and $\hat{f}$ envelope a discrete Ribaucour congruence of 2-spheres* $s_1 \wedge s_2 : \mathbb{Z}^2 \to \Lambda^2(I\!R_1^6)$, *i.e. any two corresponding elementary quadrilaterals of f and $\hat{f}$ lie on a two sphere* $S_{m,n}^2 = (S_1)_{m,n} \cap (S_2)_{m,n} \subset I\!H P^1$.

As a consequence, the Darboux transformation is involutive: since (17) is symmetric in f and $\hat{f}$, f is a Darboux transform of $\hat{f}$ as soon as $\hat{f}$ is known to be isothermic with the same cross ratio as f. Thus, any isothermic net together with a D-transform $\hat{f}$ of f form a "Darboux pair" of isothermic nets.

Permutability of different D-transforms is, again, a direct consequence of the hexahedron lemma (cf.[16]):

Theorem. *Given two D-transforms* $\hat{f}_1 = D_{\lambda_1} f$ *and* $\hat{f}_1 = D_{\lambda_1} f$ *of an isothermic net* $f : \mathbb{Z}^2 \to I\!H P^1$, *there is an isothermic net* $\hat{\hat{f}} = D_{\lambda_2} \hat{f}_1 = D_{\lambda_1} \hat{f}_2$: *in this sense,* $D_{\lambda_1} D_{\lambda_2} = D_{\lambda_2} D_{\lambda_1}$; $\hat{\hat{f}}$ *is given by the relation* $[f, \hat{f}_2, \hat{\hat{f}}, \hat{f}_1] \equiv \frac{\lambda_1}{\lambda_2}$.

3.6 Permutability

theorems also control the interrelations of different transformations of isothermic nets: all the various transformations are related by certain "commutation relations". We start with the permutability of the C- and D-transformations in Euclidean space, giving a proof similar to the one in [16]:

Theorem. *The C-transforms of a D_λ-pair $\mathfrak{f}, \hat{\mathfrak{f}} : \mathbb{Z}^2 \to \mathbb{H}$ of isothermic nets form (if properly scaled and positioned) a D_λ-pair of isothermic nets: $D_\lambda C = C D_\lambda$*

Proof. Let $\mathfrak{f}^*$ be a C-transform of $\mathfrak{f}$, and $\tilde{\mathfrak{f}} := \mathfrak{f}^* + \frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}$. Then, (16) yields

$$
\begin{array}{rcl}
(\partial_1 \tilde{\mathfrak{f}})_{m,n} & = & \lambda[\frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}]_{m,n} (\partial_1 \mathfrak{f})_{m,n} [\frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}]_{m+1,n}, \\
(\partial_2 \tilde{\mathfrak{f}})_{m,n} & = & \lambda[\frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}]_{m,n} (\partial_2 \mathfrak{f})_{m,n} [\frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}]_{m,n+1},
\end{array}
$$

i.e. $\tilde{\mathfrak{f}}$ is a D_λ-transform of $\mathfrak{f}^*$. On the other hand, using the symmetries of the cross ratio, $(\partial_1 \tilde{\mathfrak{f}})(\partial_1 \hat{\mathfrak{f}}) = (\partial_1 \mathfrak{f})(\partial_1 \mathfrak{f}^*)$ and $(\partial_2 \tilde{\mathfrak{f}})(\partial_2 \hat{\mathfrak{f}}) = (\partial_2 \mathfrak{f})(\partial_2 \mathfrak{f}^*)$ such that $\tilde{\mathfrak{f}} = \hat{\mathfrak{f}}^*$ also is a C-transform of $\hat{\mathfrak{f}}$.   ◁

This permutability theorem can obviously be read the other way: given a Christoffel pair $\mathfrak{f}, \mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$ of isothermic nets, then any suitably attuned Darboux transforms form a Christoffel pair, again.

The following gives the effect of the T-transformation on Christoffel pairs:

Lemma. *Let $\mathfrak{f}, \mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$ be a Christoffel-pair of isothermic nets; $F^\lambda = T^\lambda F^0$ a solution of the system (8). Then, $F^\lambda v_\infty = (T^*)^\lambda (v_0 + v_\infty \mathfrak{f}^*)$ is a T-transform of $\mathfrak{f}^* \simeq (v_0 + v_\infty \mathfrak{f}^*)$ where $(T^*)^\lambda = T^\lambda (v_\infty \nu_\infty + \lambda(v_0 + v_\infty \mathfrak{f})(\nu_0 - \mathfrak{f}^* \nu_\infty))$.*

Proof. It is straightforward to check that $F^\lambda (v_\infty \nu_\infty + \lambda v_0 \nu_0)$ satisfies (8) with the roles of $\mathfrak{f}$ and $\mathfrak{f}^*$ interchanged; hence, $F^\lambda (v_\infty \nu_\infty + \lambda v_0 \nu_0) v_0 = F^\lambda v_\infty$ is a T-transform of $(v_0 + v_\infty \mathfrak{f}^*)$. With the Euclidean frame $id + v_\infty \mathfrak{f}^* \nu_\infty$ of $\mathfrak{f}^*$, $(T^*)^\lambda = F^\lambda (v_\infty \nu_\infty + \lambda v_0 \nu_0)(id - v_\infty \mathfrak{f}^* \nu_\infty) = T^\lambda (v_\infty \nu_\infty + \lambda(v_0 + v_\infty \mathfrak{f})(\nu_0 - \mathfrak{f}^* \nu_\infty))$ since, up to Möbius transformation, the solutions of (8) and (7) are related by the Euclidean frame of the underlying isothermic surface.   ◁

Corollary. *The T transforms $T^\lambda (v_0 + v_\infty \mathfrak{f})$ and $(T^*)^\lambda (v_0 + v_\infty \mathfrak{f}^*)$ of a Christoffel pair $\mathfrak{f}, \mathfrak{f}^* : \mathbb{Z}^2 \to \mathbb{H}$ form (if properly positioned) a Darboux pair: $T^\lambda C = D_{-\lambda} T^\lambda$.*

Proof. From the previous lemma, $(T^*)^\lambda (v_0 + v_\infty \mathfrak{f}^*) = T^\lambda v_\infty$ with the "proper positioning" $(T^*)^\lambda = T^\lambda (v_\infty \nu_\infty + \lambda(v_0 + v_\infty \mathfrak{f})(\nu_0 - \mathfrak{f}^* \nu_\infty))$ such that the claim follows since $(T^\lambda)^{-1} = T^{-\lambda}$.   ◁

The converse of this statement is true, too: any Darboux pair of isothermic nets comes from a Christoffel pair via T-transformation:

Theorem. *The T-transforms $T^\lambda f$ and $\hat{T}^\lambda \hat{f}$ of a D_λ-pair $f, \hat{f} : \mathbb{Z}^2 \to \mathbb{H}P^1$ form (after proper stereographic projection) a Christoffel pair: $C T^\lambda = T^\lambda D_\lambda$.*

Proof. Choose homogeneous coordinates $v_0, v_\infty \in \mathbb{H}^2$ of two points in $\mathbb{H}P^1$. As $\hat{f}$ is a D_λ-transform of f, without loss of generality[5] $T^\lambda \hat{f} \equiv v_\infty \in \mathbb{H}^2$, and

---

[5] $T^\lambda \hat{f} \equiv const \in \mathbb{H}P^1$: thus, add a Möbius transformation to T^λ to obtain $T^\lambda \hat{f} \equiv v_\infty \in \mathbb{H}P^1$ and rescale $\hat{f} \to \hat{f}(\nu_0 T^\lambda \hat{f})^{-1}$; then, rescale $f \to f(\nu_\infty T^\lambda f)^{-1}$.

$T^\lambda f = v_0 + v_\infty \mathfrak{f}$ with $\mathfrak{f} : \mathbb{Z}^2 \to I\!\!H$. In this setting, $f = T^{-\lambda}(v_0 + v_\infty \mathfrak{f})$ and $\hat{f} = T^{-\lambda} v_\infty$ with the T-transform $T^{-\lambda} : \mathbb{Z}^2 \to PGl(2, I\!\!H)$ associated to $v_0 + v_\infty \mathfrak{f}$ and its C-transform $\mathfrak{f}^* \simeq v_0 + v_\infty \mathfrak{f}^*$. But, $T^{-\lambda} v_\infty = (T^*)^{-\lambda}(v_0 + v_\infty \mathfrak{f}^*)$ is the $T^{-\lambda}$-transform of $\mathfrak{f}^*$; thus, up to Möbius transformation $\hat{T}^\lambda = ((T^*)^{-\lambda})^{-1}$, and hence $\hat{T}^\lambda \hat{f} = v_0 + v_\infty \mathfrak{f}^*$. Consequently, with this positioning of $\hat{T}^\lambda \hat{f}$, both nets, $T^\lambda f$ and $\hat{T}^\lambda \hat{f}$, simultanously project to a C-pair.   ◁

As an immediate consequence of the previous theorem, we see that the "difference" between two D-transforms can be "measured" by the G-transformation:

Corollary. *The T-transforms $\hat{T}_1^\lambda \hat{f}_1$ and $\hat{T}_2^\lambda \hat{f}_2$ of two (different) D_λ-transforms $\hat{f}_1, \hat{f}_2 : \mathbb{Z}^2 \to I\!\!HP^1$ of an isothermic net $f : \mathbb{Z}^2 \to I\!\!HP^1$ are (after proper stereographic projections) G-transforms of each other.*

Another consequence of the interrelations of D- and C-pairs under the T-transformation stated above is the "generic" behaviour of the Darboux pairs under the T-transformation:

Corollary. *The T-transforms $T^\mu f$ and $\hat{T}^\mu \hat{f}$, $\mu \neq \lambda$, of a D_λ-pair of isothermic nets, $f, \hat{f} : \mathbb{Z}^2 \to I\!\!HP^1$, form (if properly positioned) a $D_{\lambda-\mu}$-pair: in this sense, $T^\mu D_\lambda = D_{\lambda-\mu} T^\mu$. Moreover, $\hat{T}^\mu = T^\mu(1 - \frac{\mu}{\lambda} f(\hat{\varphi} f)^{-1} \hat{\varphi})$.*

Proof. By the previous theorem, $f = T^{-\lambda}(v_0 + v_\infty \mathfrak{f})$ and $\hat{f} = (T^*)^{-\lambda}(v_0 + v_\infty \mathfrak{f}^*)$ for a suitable C-pair $\mathfrak{f}, \mathfrak{f}^* : \mathbb{Z}^2 \to I\!\!H$; with $T^\mu T^{-\lambda} = T^{\mu-\lambda}$, this implies the first statement. The second statement follows by a straightforward computation: $(T^*)^{\mu-\lambda}((T^*)^\lambda)^{-1} = T^\mu(1 - \frac{\mu}{\lambda} T^{-\lambda}(v_0 + v_\infty \mathfrak{f}) \nu_\infty (T^{-\lambda})^{-1})$.   ◁

Combining the above permutability theorems on the effect of the D- and T-transformations on Christoffel pairs, a larger permutability scheme is obtained:

Theorem. *The T^λ-transformation preserves the C-D_λ-permutability:*
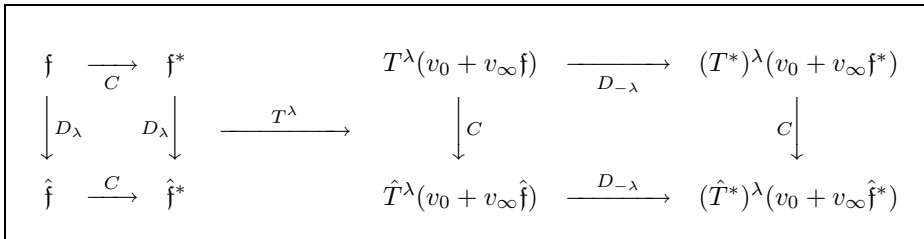

Fig. 2. A permutability theorem

Proof. Given four isothermic nets $\mathfrak{f}, \hat{\mathfrak{f}}, \hat{\mathfrak{f}}^*, \mathfrak{f}^* : \mathbb{Z}^2 \to I\!\!H$ that form two D_λ-pairs and two C-pairs according to the C-D_λ-permutability theorem, two $D_{-\lambda}$-pairs

$T^\lambda(v_0 + v_\infty\mathfrak{f}), (T^*)^\lambda(v_0 + v_\infty\mathfrak{f}^*)$ and $\hat{T}^\lambda(v_0 + v_\infty\hat{\mathfrak{f}}), (\hat{T}^*)^\lambda(v_0 + v_\infty\hat{\mathfrak{f}}^*)$ are obtained via suitably adjusted T-transformations:

$$
\begin{array}{rcl}
(T^*)^\lambda & = & T^\lambda(v_\infty\nu_\infty + \lambda(v_0 + v_\infty\mathfrak{f})(\nu_0 - \mathfrak{f}^*\nu_\infty)), \\
(\hat{T}^*)^\lambda & = & \hat{T}^\lambda(v_\infty\nu_\infty + \lambda(v_0 + v_\infty\hat{\mathfrak{f}})(\nu_0 - \hat{\mathfrak{f}}^*\nu_\infty)).
\end{array}
$$

On the other hand, $T^\lambda(v_0 + v_\infty\hat{\mathfrak{f}}) = (T^*)^\lambda(v_0 + v_\infty\hat{\mathfrak{f}}^*) \in I\!\!H P^1$ define the same point in $I\!\!H P^1$ since $(\hat{\mathfrak{f}}^* - \mathfrak{f}^*) = \frac{1}{\lambda}(\hat{\mathfrak{f}} - \mathfrak{f})^{-1}$; hence, $T^\lambda(v_0 + v_\infty\mathfrak{f}), \hat{T}^\lambda(v_0 + v_\infty\hat{\mathfrak{f}})$ and $(T^*)^\lambda(v_0 + v_\infty\mathfrak{f}^*), (\hat{T}^*)^\lambda(v_0 + v_\infty\hat{\mathfrak{f}}^*)$ simultanously project to C-pairs, when, additionally, T^λ and $\hat{T}^\lambda$ (resp. $(T^*)^\lambda$ and $(\hat{T}^*)^\lambda$) are "properly adjusted", as in the proof of the $(CT^\lambda = T^\lambda D_\lambda)$-permutability theorem. $\triangleleft$

Having established this transformation theory for discrete isothermic nets, we are prepared to study

4. Horospherical nets in hyperbolic space

as a discrete analog of surfaces of constant mean curvature 1 — the mean curvature of horospheres — in hyperbolic space. Smooth cmc-1 surfaces in hyperbolic space appear as T-transforms[6] (cousins) of minimal surfaces in Euclidean space, and, on the other hand, as D-transforms of their hyperbolic Gauss maps [15]. These facts suggest two different ansatzes to define discrete horospherical nets: as T-transforms of discrete minimal nets, or as D-transforms of discrete isothermic nets in a 2-sphere. Both ansatzes will turn out to give the same class of discrete nets.

First, recall the definition of discrete minimal nets [5]:

Definition (Minimal nets). *An isothermic net* $\mathfrak{f} : Z^2 \to \mathrm{Im}I\!\!H$ *is called minimal if it is a C-transform* $\mathfrak{f} = \mathfrak{n}^*$ *of an isothermic net* $\mathfrak{n} : Z^2 \to S^2 \subset \mathrm{Im}I\!\!H$ *into the 2-sphere;* $\mathfrak{n}$ *is then called the Gauss map of the discrete minimal net.*

As an immediate consequence of this definition, discrete minimal nets can be obtained from complex valued isothermic nets ("discrete holomorphic nets"): the stereographic projection $i(i + \mathfrak{g}j)(i - \mathfrak{g}j)^{-1} : \mathbb{Z}^2 \to S^2$ of any discrete isothermic net $\mathfrak{g}j : \mathbb{Z}^2 \to \mathbb{C}j \cong \mathbb{C}$ appears as a Gauss map (or, C-transform) of a discrete minimal net — that, therefore, can be (re-) constructed from $\mathfrak{g} : \mathbb{Z}^2 \to \mathbb{C}$. A discrete analog of the classical Weierstrass representation for minimal surfaces is obtained via the Goursat transformation: given an isothermic net $\mathfrak{h} : \mathbb{Z}^2 \to \mathbb{C}$, a minimal net $\mathfrak{f} : \mathbb{Z}^2 \to \mathrm{Im}I\!\!H$ is obtained as the C-transform of the stereographic

---

[6] Recall that, in case of constant mean curvature surfaces in space forms, the T-transformation becomes transformation via Lawson correspondence; for minimal surfaces in Euclidean space, the T-transformation provides the Umehara-Yamada perturbation.

projection of[7] $\mathfrak{g} = \mathfrak{h}^*$ — with $\tilde{v}_0 = (v_0 - v_\infty i)\frac{i}{\sqrt{2}}$ and $\tilde{v}_\infty = (v_0 + v_\infty i)\frac{1}{\sqrt{2}}$, (15) yields a discrete version of the (quaternionic: cf.[15]) Weierstrass formula [6]:

$$
\begin{array}{rcl}
(\partial_1 \mathfrak{f})_{m,n} &=& \frac{1}{2}(i - \mathfrak{g}j)_{m,n} j (\partial_1 \mathfrak{h})_{m,n} (i - \mathfrak{g}j)_{m+1,n}, \\
(\partial_2 \mathfrak{f})_{m,n} &=& \frac{1}{2}(i - \mathfrak{g}j)_{m,n} j (\partial_2 \mathfrak{h})_{m,n} (i - \mathfrak{g}j)_{m,n+1}.
\end{array}
$$

Another direct consequence of this definition for discrete minimal nets is the promised equivalence of our two ansatzes to define a discrete analog of constant mean curvature 1 surfaces in hyperbolic space:

Lemma. *An isothermic net is a T-transform of a discrete minimal net iff it is a D-transform of an isothermic net on the 2-sphere.*

Proof. This is a consequence of two permutability theorems: a discrete minimal net $\mathfrak{f} : \mathbb{Z}^2 \to \mathrm{Im}\,\mathbb{H}$ and its Gauss map $\mathfrak{n} : \mathbb{Z}^2 \to S^2$ form a C-pair, thus their $T^{-\lambda}$-transforms $T^{-\lambda}(v_0 + v_\infty \mathfrak{f}) : \mathbb{Z}^2 \to S^3$ and $(T^*)^{-\lambda}(v_0 + v_\infty \mathfrak{n}) : \mathbb{Z}^2 \to S^2$ form (if properly positioned) a D_λ-pair, $T^{-\lambda}C = D_\lambda T^{-\lambda}$; on the other hand, the T^λ transforms of a D_λ-pair $f : \mathbb{Z}^2 \to S^3$ and $n : \mathbb{Z}^2 \to S^2$ form (after proper stereographic projection) a C-pair, $T^\lambda D_\lambda = CT^\lambda$, and $T^\lambda n : \mathbb{Z}^2 \to S^2$ takes values in some 2-sphere while $T^\lambda f : \mathbb{Z}^2 \to \mathrm{Im}\,\mathbb{H}$.   ◁

Thus, similar to the definition of minimal nets, we define a discrete analog for constant mean curvature 1 surfaces in hyperbolic space via coupling with their hyperbolic Gauss maps. This determines $H^3 \subset S^3$ as one of the connected components of $S^3 \setminus S^2$ of the complement of its infinity boundary $S^2 \cong \partial H^3$:

Definition (Horospherical nets). *An isothermic net $f : \mathbb{Z}^2 \to S^3 \setminus S^2$ is called horospherical if it is a D-transform of an isothermic net $n : \mathbb{Z}^2 \to S^2 \cong \partial H^3$; this net, n, is then called the hyperbolic Gauss map of f.*

This definition directly leads to a representation in terms of "discrete holomorphic" data: given a discrete isothermic net $\mathfrak{g} : \mathbb{Z}^2 \to \mathbb{C}$ together with its C-transform $\mathfrak{h} := \mathfrak{g}^* : \mathbb{Z}^2 \to \mathbb{C}$ (i.e. $(\mathfrak{g}j)^* = -j\mathfrak{h}$), any D-transform of $n = v_0 + v_\infty \mathfrak{g}j$ comes from a solution $T^\lambda : \mathbb{Z}^2 \to Gl(2, \mathbb{H})$ of (T),

$$
\begin{array}{rcl}
T^\lambda_{m+1,n} &=& T^\lambda_{m,n}(1 - \lambda[v_0 + v_\infty \mathfrak{g}_{m,n}j]j(\partial_1 \mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m+1,n}j\nu_\infty]) \\
&=& T^\lambda_{m,n}J^{-1}(1 + \lambda[v_0 + v_\infty \mathfrak{g}_{m,n}](\partial_1 \mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m+1,n}\nu_\infty])J, \\
T^\lambda_{m,n+1} &=& T^\lambda_{m,n}(1 + \lambda[v_0 + v_\infty \mathfrak{g}_{m,n}j]j(\partial_2 \mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m,n+1}j\nu_\infty]) \\
&=& T^\lambda_{m,n}J^{-1}(1 - \lambda[v_0 + v_\infty \mathfrak{g}_{m,n}](\partial_2 \mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m,n+1}\nu_\infty])J,
\end{array}
\qquad (18)
$$

via $D_\lambda n = (T^{-\lambda})^{-1}p_0$ where $p_0 \in \mathbb{H}P^1$ is a (constant) point. Here, we denoted $J := (v_\infty \nu_0 + v_0 j\nu_\infty)$: obviously, (18) is a *complex* system for $T^\lambda J^{-1}$ — indeed,

---

[7] Note, that we are working with "principal coordinates" so that the minimal surface is already determined by just *one* holomorphic function.

it is a discrete version of the system arising in Bryant's Weierstrass type representation for constant mean curvature 1 surfaces in hyperbolic space [20]. Thus, $f^\sharp = (T^{-\lambda})^{-1}p_0$ is a horospherical net — with hyperbolic Gauss map n — if and only if $f^\sharp : \mathbb{Z}^2 \to S^3 \setminus (\mathbb{C}j \cup \{\infty\})$, i.e. if $(T^{-\lambda})^{-1}_{m,n}p_0 \notin \mathbb{C}j \cup \{\infty\}$ at *one* point (m, n). Clearly, *any* horospherical net can be (re-) constructed from its hyperbolic Gauss map this way:

Theorem. *Let $\mathfrak{g}, \mathfrak{h} : \mathbb{Z}^2 \to \mathbb{C}$ be a Christoffel pair of discrete isothermic nets, let $\mathfrak{p}_0 \in \mathrm{Im}I\!H \setminus \mathbb{C}j$, and $\tau^\lambda : \mathbb{Z}^2 \to Gl(2, I\!H)$, $\tau^\lambda_{0,0} = id$, be a solution of*

$$
\begin{array}{rcl}
\tau^\lambda_{m+1,n} & = & \tau^\lambda_{m,n}(1 + \lambda[v_0 + v_\infty\mathfrak{g}_{m,n}](\partial_1\mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m+1,n}\nu_\infty]), \\
\tau^\lambda_{m,n+1} & = & \tau^\lambda_{m,n}(1 + \lambda[v_0 + v_\infty\mathfrak{g}_{m,n}](\partial_2\mathfrak{h})_{m,n}[\nu_0 - \mathfrak{g}_{m,n+1}\nu_\infty]).
\end{array} \tag{H}
$$

Then, $f^\sharp = (v_\infty\nu_0 - v_0 j\nu_\infty)(\tau^{-\lambda})^{-1}(v_0 + v_\infty\mathfrak{p}_0) : \mathbb{Z}^2 \to S^3 \setminus S^2$ is a horospherical net with hyperbolic Gauss map $n = v_0 + v_\infty\mathfrak{g}j : \mathbb{Z}^2 \to S^2 = \mathbb{C}j \cup \{\infty\}$.

Every horospherical net $f^\sharp : \mathbb{Z}^2 \to S^3 \setminus (\mathbb{C}j \cup \{\infty\})$ can be constructed this way.

On the other hand, horospherical nets are T-transforms of minimal nets in Euclidean 3-space $\mathrm{Im}I\!H$: with the discussion above, on the discrete Weierstrass representation for minimal nets, and the $(T^\lambda C = D_{-\lambda}T^\lambda)$-permutability theorem, we conclude that any solution T^λ of (18) provides a horospherical net via $f = T^\lambda(v_0 + v_\infty i)\frac{1}{\sqrt{2}}$ — recall that (18) (or, (T)) does not depend on the stereographic projection of $n = v_0 + v_\infty\mathfrak{g}j$ used to define the C-transform. Thus,

Theorem. *Let $\mathfrak{g} : \mathbb{Z}^2 \to \mathbb{C}$ be an isothermic net, τ^λ, $\lambda \neq 0$, a solution of (H) as above; then, $f = (v_\infty\nu_0 - v_0 j\nu_\infty)\tau^\lambda(v_\infty i + v_0 j)\frac{1}{\sqrt{2}}$ is a horospherical net with hyperbolic Gauss map $n^\lambda = (v_\infty\nu_0 - v_0 j\nu_\infty)\tau^\lambda(v_0 + v_\infty\mathfrak{g})j : \mathbb{Z}^2 \to S^2 = \mathbb{C}j \cup \{\infty\}$.*

Note, that up to a change of model for the hyperbolic space, this is the *exact* discrete analog of Bryant's Weierstrass type representation for cmc-1 surfaces in hyperbolic space [20].

Definition. *We refer to $\mathfrak{g}$ as the secondary Gauss map of the horospherical net f, and to the minimal net $[i(i + \mathfrak{g}j)(i - \mathfrak{g}j)^{-1}]^*$ as the minimal cousin of f.*

As an example, Figure 1 indicates the family of horospherical nets with the discrete catenoid (middle picture in figure 1) as minimal cousins, obtained from the discrete version of Bryant's Weierstrass type representation: the "discrete holomorphic nets" $\mathfrak{g}_{m,n} := e^{2\pi\frac{m+in}{N}}$ and $\mathfrak{h}_{m,n} = e^{-2\pi\frac{m+in}{N}}$ form a Christoffel pair since, by a straightforward calculation, the (constant) cross ratios of elementary quadrilaterals of $\mathfrak{g}$ satisfy $q = \frac{\partial_1\mathfrak{h}\partial_1\mathfrak{g}}{\partial_2\mathfrak{h}\partial_2\mathfrak{g}}$.

Since the "difference" between two D_λ-transforms of an isothermic net is measured by a G-transformation, the minimal cousins of two horospherical nets with the same hyperbolic Gauss map are G-transforms of each other.

The T-transforms of an isothermic net $n : \mathbb{Z}^2 \to S^2$ take values in S^2. Hence, the $(T^\mu D_\lambda = D_{\lambda-\mu}T^\mu)$-permutability theorem shows that (regular) T-transforms

of horospherical nets are horospherical. Moreover, all regular T-transforms of a horospherical net have the same minimal cousin by the "1-parameter group property" of the T-transformation, $T^{\lambda_2}T^{\lambda_1} = T^{\lambda_1+\lambda_2}$.

In [16], we showed that discrete minimal nets allow ∞^3 D-transforms into minimal nets; this result carries over for horospherical nets via the T-transformation.

By the compatibility of the C-D_λ-permutability with the T^λ-transformation (see figure 2), the roles of the secondary and hyperbolic Gauss maps are interchanged for the horospherical nets $f^\sharp$ and f obtained from a "holomorphic net" $\mathfrak{g}$ by the two different representations given above, $n^\lambda = n^\sharp$; therefore, f and $f^\sharp$ can be considered as "dual" horospherical nets (cf.[21]).

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